Tutorial 9

Decomposition of state space for periodic MC

Let X_n, $n \geq 0$, be an irreducible Markov chain with state space S. The *period* is given by

$$d = \gcd\{n \geq 1 : P^n(x,x) > 0\}, \qquad x \in S,$$

where gcd means the *greatest common divisor* and d is independent of the choice of x. Notice that it may be not true that $P^d(x,x) > 0$. Refer to Exercise 22 in Chapter 2 of the textbook for an example.

Suppose that the chain is periodic ($d \geq 2$). Let $Y_m = X_{md}$, $m \geq 0$. Then Y_m, $m \geq 0$, is an aperiodic Markov chain with transition matrix $Q = P^d$, since the period is $\gcd\{m \geq 1 : Q^m(x,x) = P^{md}(x,x) > 0\} = \dfrac{d}{d} = 1$. Moreover, the chain $\{Y_m\}$ is reducible and S is a union of d irreducible closed sets:

$$S = C_1 \cup C_2 \cup \cdots \cup C_d, \tag{1}$$

where for $i \in \{1, 2, \ldots, d\}$ and $x \in C_i$ (we set $C_{d+1} = C_1$ here) we have

$$P(x,y) > 0 \quad \text{only if} \quad y \in C_{i+1}.$$

Example. $S = \{1, 2\}$.

$$P = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ & \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \end{matrix}$$

Then $P^2 = I$ with period $d = 2$ and $S = C_1 \cup C_2 = \{1\} \cup \{2\}$. It has a stationary distribution $\pi = (1/2, 1/2)$.

Example. $S = \{1, 2, 3, 4\}$,

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{matrix}.$$

Observe that there are two '3-circles': $1 \to 2 \to 3 \to 1$ and $2 \to 3 \to 4 \to 2$. Hence any two states can communicate, so the chain is irreducible.

Let $\pi = (\pi(1), \pi(2), \pi(3), \pi(4))$ be the stationary distribution. Then $\pi P = \pi$ implies that

$$\begin{cases} \frac{2}{3}\pi(3) = \pi(1), \\ \pi(1) + \pi(4) = \pi(2), \\ \pi(2) = \pi(3), \\ \frac{1}{3}\pi(3) = \pi(4). \end{cases}$$

Together with $\pi(1) + \pi(2) + \pi(3) + \pi(4) = 1$, we get $\pi = (\pi(1), \pi(2), \pi(3), \pi(4)) = \left(\frac{2}{9}, \frac{1}{3}, \frac{1}{3}, \frac{1}{9}\right)$.

Note that

$$P^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad P^3 = \begin{pmatrix} \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad P^4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 \end{pmatrix} = P.$$

Inductively we have $P^{3k-2} = P$, $P^{3k-1} = P^2$, and $P^{3k} = P^3$ for all $k \geq 1$. Hence the limit $\lim_{k\to\infty} P^k$ does not exists even the chain has a unique stationary distribution.

Remark. It can be checked that there are two eigenvalues $\frac{1}{2}(-1 \pm i\sqrt{3})$ of P whose moduli is not less than 1 i.e. it does not satisfy the third assumption in theorem 2 of tutorial note 7.

Easily we get the period $d = 3$ and $S = C_1 \cup C_2 \cup C_3$ with

$$C_1 = \{1, 4\}, \quad C_2 = \{2\}, \quad C_3 = \{3\}.$$

Convergence to the stationary distribution: periodic case

Now we continue to study the limit behavior of a MC, and more precisely, to study the relation between the limit of $P^k(x, y)$ and the stationary distribution.

In Tutorial 6, we discussed some irreducible cases under 3 assumptions (actually implies the aperiodicity), so that the stationary distribution π exists and each row of the limit matrix $\lim_{k\to\infty} P^k$ is π. Moreover, a basic theorem says that (see textbook, aperiodic case in Theorem 7 in P73):

Theorem 1. If a MC is irreducible, positive recurrent and aperiodic, then the stationary distribution π uniquely exists and

$$\lim_{k\to\infty} P^k(x, y) = \pi(y) = \frac{1}{m_y}, \tag{2}$$

where $m_y = E_y(T_y)$ is the mean return time to y.

Also in Tutorial 6, we learnt a method to calculate the limit matrix $\lim_{k\to\infty} P^k$ when the chain is reducible and each block of an irreducible closed set satisfies the 3 assumptions. So all the chains we have discussed are still aperiodic.

Until now we know nothing about the periodic case. Actually in periodic case, $\lim_{k\to\infty} P^k$ does not exist even if the chain has a unique stationary distribution. But for the limit of subsequence, the following theorem is really crucial:

Theorem 2. Let X_n, $n \geq 0$, be an irreducible positive recurrent MC having stationary distribution π. If the chain is periodic with period d, then for any $x, y \in S$, there is an integer r, $0 \leq r < d$, such that $P^n(x, y) > 0$ only if $n = md + r$ for some $m \in \mathbb{N}$, and

$$\lim_{m\to\infty} P^{md+r}(x, y) = d \cdot \pi(y). \tag{3}$$

For $x \in C_i$ and $y \in C_j$,

$$r = \begin{cases} j - i, & \text{if } i \leq j, \\ j + d - i, & \text{if } i > j. \end{cases}$$

Proof. We prove this through the following three steps.

Step 1. Consider the case of $x = y$.

For any $x \in \mathcal{S}$, by the definition of period ($d = g.c.d.\{n \geq 1 : P^n(x,x) > 0\}$),

$$P^n(x,x) > 0 \quad \text{only if} \quad n = md \;\text{ for some } m \in \mathbb{N}.$$

Let $Y_m = X_{md}$, $m \geq 0$. Then follows (1) we have a decomposition of $\mathcal{S}$. Suppose that x belongs to some irreducible closed set $\mathcal{C}_i$.

Let $X_0 = Y_0 = x$, then the mean return time to x with respect to Y_m is m_x/d, where m_x is the mean return time to x with respect to X_n. If we restrict the chain Y_m to a smaller state space $\mathcal{C}_i$, then the new MC is irreducible, positive recurrent and aperiodic. By Theorem 1, the new chain has a unique stationary distribution π_i and

$$\lim_{m\to\infty} Q^m(x,x) = \pi_i(x) = \frac{1}{m_x/d} = d/m_x = d\pi(x).$$

Hence for any $x \in \mathcal{S}$,

$$\lim_{m\to\infty} P^{md}(x,x) = d\pi(x). \tag{4}$$

Step 2. Find r for general x, y.

Let $x, y \in \mathcal{S}$, and $r_1 = \min\{n \geq 1 : P^n(x,y) > 0\}$.

By irreducibility, we can choose n_1 such that $P^{n_1}(y,x) > 0$. Then

$$P^{r_1+n_1}(x,x) \geq P^{r_1}(x,y)P^{n_1}(y,x) > 0$$

which implies that d is a divisor of $r_1 + n_1$. If n satisfies $P^n(x,y) > 0$, then

$$P^{n+n_1}(x,x) \geq P^n(x,y)P^{n_1}(y,x) > 0$$

so that d is also a divisor of $n + n_1$. As a result, d is a divisor of $n - r_1$.

Let $r_1 = m_1 d + r$, where $0 \leq r < d$, then

$$P^n(x,y) > 0 \quad \text{only if} \quad n = md + r \;\text{ for some } m \in \mathbb{N}.$$

Remark. One can check that for $x \in \mathcal{C}_i$ and $y \in \mathcal{C}_j$,

$$r = \begin{cases} j - i, & \text{if } i \leq j, \\ j + d - i, & \text{if } i > j. \end{cases}$$

Step 3. Prove formula (3).

Now we can write

$$\begin{aligned} P^{md+r}(x,y) &= \sum_{j=1}^{md+r} P_x(T_y = j)P^{md+r-j}(y,y) \\ &= \sum_{k=0}^{m} P_x(T_y = kd + r)P^{(m-k)d}(y,y). \end{aligned}$$

Apply the bounded convergence theorem (in tutorial 7) to

$$a_m(k) = \begin{cases} P^{(m-k)d}(y,y), & 0 \leq k \leq m, \\ 0. & k > m, \end{cases}$$

and $p_k = P_x(T_y = kd + r)$, then by (4),

$$\begin{aligned}
\lim_{m\to\infty} P^{md+r}(x,y) &= \lim_{m\to\infty}\sum_{k=0}^{\infty} P_x(T_y = kd+r)P^{(m-k)d}(y,y) \\
&= \sum_{k=0}^{\infty} P_x(T_y = kd+r)\lim_{m\to\infty} P^{(m-k)d}(y,y) \\
&= \sum_{k=0}^{\infty} d\pi(y)P_x(T_y = kd+r) \\
&= d\pi(y).
\end{aligned}$$

□

Example. Consider the Ehrenfest chain with $d = 3$. The transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The state space $S = \{0,1,2,3\}$, we have $C_1 = \{0,2\}, C_2 = \{1,3\}$. Obviously the period of the chain is 2. By direct calculation, the chain has a unique stationary distribution $\pi = \left(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\right)$. Simply applying Theorem 2, we have

$$\lim_{k\to\infty} P^{2k} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \end{pmatrix} \quad \text{and} \quad \lim_{k\to\infty} P^{2k+1} = \begin{pmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \end{pmatrix}.$$